Polynomial-time Algorithm for Computing the Yolk in Fixed Dimension

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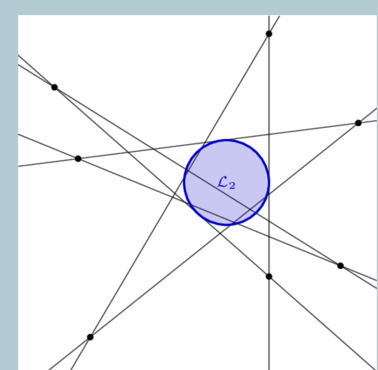
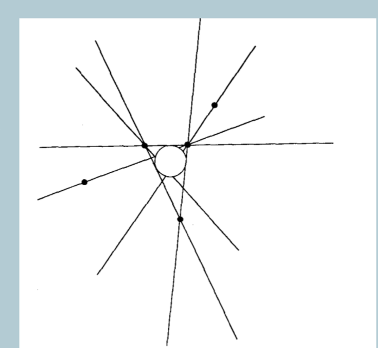
Abstract



(Joachim 2012)



(the yolk)

The yolk is an important concept in spatial voting games, it is the ball of minimum radius that intersects all median hyperplanes. A median line hyperplane is any plane that divides the plane into two closed half space each with at most half amount of all points. The position of the yolk shows the approximate center of the distribution of voters, while the magnitude of the yolk's size illustrates the deviation from one that would result in a majority rule equilibrium or core outcome.

The yolk plays a pivotal role in the context of spatial voting games owing to its inherent simplicity and its interrelation with other theoretical constructs. Its radius offers approximation for delineating uncovered set, characterizing ε-core and etc. Therefore, by calculating the yolk's dimension, we can having a better understanding of associated theoretical notions.

Prof. Craig derives a polynomial-time algorithm to compute the yolk for any fixed dimension.

Introduction

In the context of the Euclidean spatial model, where each voter's preferred policy is represented as an ideal point in a multi-dimensional space, voters tend to favor policies closer to their individual ideal points according to the Euclidean norm. This model has extensive study and practical applications. In majority rule voting within this model, a classical Nash equilibrium signifies a point where no other point is closer to more than half the voters' ideal points. While the existence of such equilibrium is supported by theory, empirical evidence, and decision-making processes, it is often not feasible due to chaotic cycles and collapses in the space. Efforts have been dedicated to finding alternative solution concepts that are more general and in line with practical scenarios.

One prominent solution concept is the "yolk," as discussed above. This yolk concept aligns with the Nash equilibrium point when it exists and has notable connections to other solution concepts like the uncovered set, Pareto set, win set, and Shapley-Owen power scores. The yolk concept possesses favorable normative properties and bounds on outcome sets. While determining the yolk can be computationally challenging, recent work has derived a polynomial algorithm for arbitrary fixed dimensions. Although this algorithm might not be practical for higher dimensions, it remains useful for most empirical studies, often conducted in two dimensions due to the substantial predictive power of this reduced dimensionality. Additionally, the algorithm can be implemented efficiently using logarithmic time and parallel processing.

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Preliminaries

Setting up notations:
The set of voter ideal points, denoted as $V \in \mathbb{R}^m$, comprises a cardinality of n individuals. For any given hyperplane h in $\mathbb{R}^m$, we designate the two closed half-spaces defined by h as h+ and h-. A hyperplane h is identified as median with respect to V if and only if each of the closed half-spaces it delineates contains at least half of the total voters, which is expressed as: $|h+ \cap V| \geq ½n$ and $|h- \cap V| \geq ½n$.

Additionally, a hyperplane {x: p* x = p_0} is deemed consistent with a median split (S, T) if S lies within h+ and T is found within the intersection of h- and V. The collective ensemble of all possible median splits is denoted as M. Proposition 1. A hyperplane is regarded as median if and only if it aligns with a specific median split (S, T) from the set M.

Determining the yolk radius

The primary challenge in devising an efficient algorithm lies in the infinite nature of median hyperplanes. To address this, our objective is to identify a critical subset of median hyperplanes, referred to as "determining" hyperplanes, that can encapsulate the yolk's properties.

Initially, the set of extremal median hyperplanes is a natural choice for consideration. An extremal hyperplane is one that contains a specific number of voter ideal points, and it is instrumental in determining the yolk radius, particularly in two-dimensional scenarios. However, this straightforward assumption can be inadequate, leading to the need to expand the determining set.

To overcome this, a more comprehensive set, both larger than the extremal set yet relatively compact, is derived. This set, while theoretically infinite, remains only slightly larger than the extremal set, striking a balance between precision and practicality. The formal statement of Theorem 1 is deferred to later in the section to facilitate the comprehension of the terminology introduced in its proof.

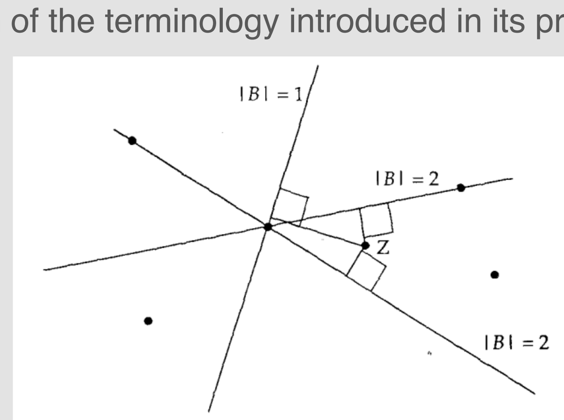

Fig. 2. Determining median lines in two dimensions.

To establish this determining set, we draw from the proof of yolk convergence outlined in a previous work. By seeking upper bounds on the yolk radius, we formulate a mathematical program that characterizes the radius. This program leverages a form of duality known as "polarity" to express median hyperplanes. For each median split, the program identifies the hyperplane farthest from a given point. The formulation evolves into a nonlinear mathematical program, providing insights into the geometric interpretation of its constraints.

To ensure an exact formulation for algorithmic purposes, the Karush-Kuhn-Tucker (KKT) conditions are applied to this mathematical program. Surprisingly, the KKT conditions yield a simplified problem, reducing consideration from an infinite to a finite collection of polynomially sized sets of hyperplanes. This transformation ultimately leads to the derivation of determining hyperplanes that possess crucial properties. These determining hyperplanes exhibit dependence on the chosen point of origin, thereby offering a nuanced understanding of their role.

The central outcome of this section is encapsulated in Theorem 1, asserting that the set of determining median hyperplanes effectively determines the radius of the yolk centered at any point. Notably, this result is translatable across different origin points, thereby extending the notion of determining hyperplanes. It is important to note that while the determining hyperplanes form a polynomially bounded set for fixed dimensions, their union is typically infinite.

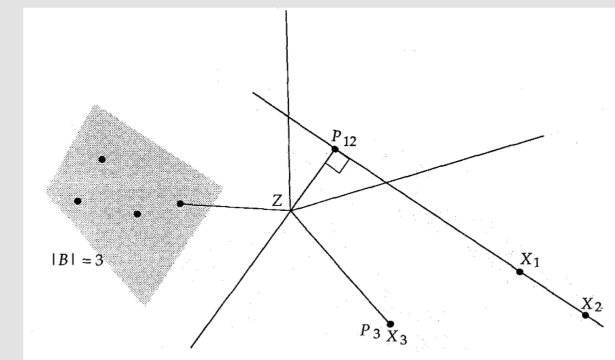

Fig. 3. Determining median hyperplanes in three dimensions.

Algorithm for yolk in fixed dimension

Then we present a polynomial algorithm for computing the yolk of a given set of points. Additionally, we demonstrate that the algorithm can operate efficiently in logarithmic time with a polynomial number of parallel processors. To establish the algorithm's foundation, we introduce a few preliminary lemmata.

To address both non-degenerate and degenerate scenarios, we introduce Lemma 2 and Theorem 2. Lemma 2 provides conditions under which a given point cannot be the center of the yolk. Specifically, it asserts that if the number of determining median hyperplanes binding at a point is less than the dimension plus one and certain conditions are met, then the point cannot be the yolk center. The proof involves demonstrating the existence of a direction that decreases the yolk radius when moving from such a point. Theorem 2 establishes a pseudo-polynomial time algorithm for computing the yolk, extending from the non-degenerate to the degenerate case. This algorithm has a time complexity of O(n(m+1)^2), where n is the number of points and m is the dimension.

Furthermore, Corollary 2 extends the algorithm's applicability to the degenerate case by showing that infinitesimal perturbations of a degenerate configuration yield infinitesimal changes in the yolk's center and radius. Corollary 3 demonstrates that the algorithm is highly parallelizable, allowing it to run in logarithmic time with a polynomial number of processors.

Despite the theoretical complexity estimates, the practical application of the algorithm may be limited due to its computational requirements. Thus, the subsequent section focuses on refining the algorithm for more feasible execution, particularly in two-dimensional scenarios of moderate size.

Lemma 1. *$r(x)$ is convex and continuous in x.*

Lemma 2. *Suppose V is nondegenerate and fewer than $m+1$ determining median hyperplanes are binding at z. If $|V|$ is odd then z is not the yolk center. In the case $|V|$ even, if moreover z is not the midpoint of the shortest line segment connecting two determining median affine hulls $\mathrm{Aff}(B_1)$, $\mathrm{Aff}(B_2)$ where $|B_1|+|B_2| \leqslant m+1$, then z is not the yolk center.*

Lemma 3. *Let V be any configuration and let $\tilde{V}$ be a nondegenerate infinitesimal perturbation of V. Then the radius and center of the yolks of V and $\tilde{V}$ differ only infinitesimally. When $|V|$ is even, the center of the yolk of $\tilde{V}$ differs only infinitesimally from one of the yolk centers of V.*

Improve in Performance and Time Bounds

An extended analysis suggests that, if a certain conjecture by Erdős, Lovász, Simmons, and Straus holds true, the worst-case complexity could be further reduced to o(n^3+ε) for any ε > 0. The algorithm's practical runtime is also estimated to be around O(n^3 log n) based on empirical observations. Additionally, a variant of the algorithm is proposed, potentially offering sub-cubic time complexity in two dimensions, although precise bounds for this remain an open research question.

The improvement involves refining the computation of equidistant points to affine hulls, particularly when a combination of lines and points is present. The revised algorithm selectively discards equidistant points that are farther away from the corresponding affine hulls, as such points are unlikely to be the yolk center. The modified approach then focuses on 3-tuples of affine hulls, computing the closest equidistant point and determining the median status of hyperplanes. The overall time complexity is estimated to be O(n m^2 + m), with considerations for preprocessing extremal hyperplanes and handling various tuple scenarios.

The discussion highlights the potential for parallelizing the algorithm and mentions the possible use of preprocessing techniques to achieve even more efficient solutions. In particular, it is suggested that by utilizing linear programming and focusing on extremal hyperplanes, a sub-cubic time complexity might be attainable. This could significantly enhance the efficiency of yolk computations, especially in higher dimensions.

Future Plans

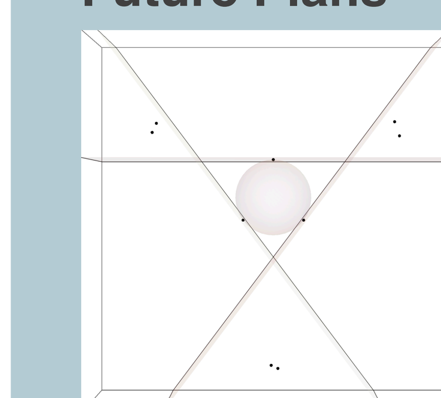

Figure 1

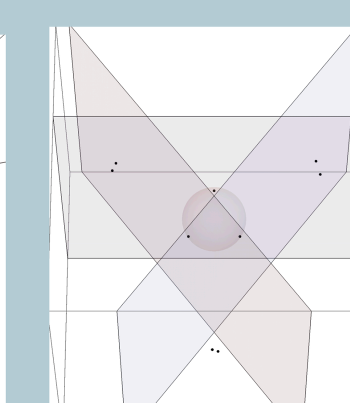

Figure 2

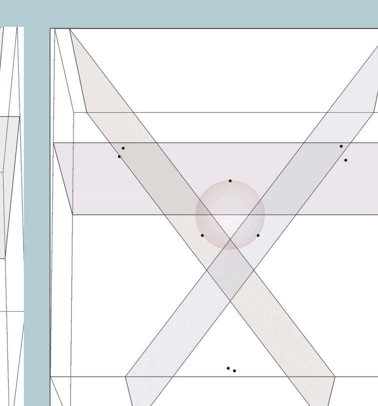

Figure 3

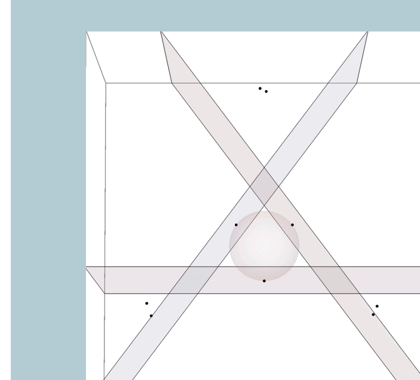

Figure 4

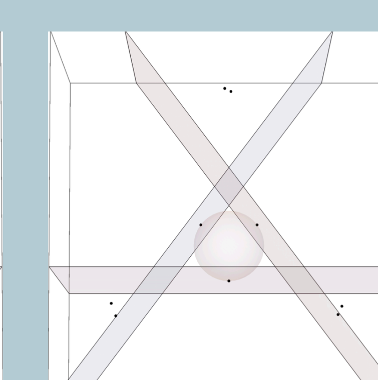

Figure 5

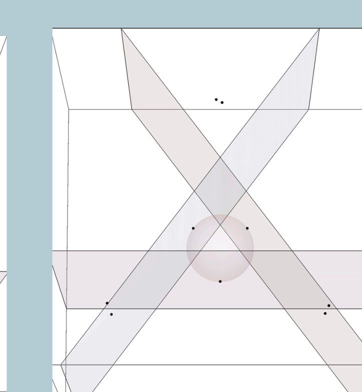

Figure 6

Figures above are 3D figures from different vantage points, that show a possible counter-example to the algorithm that already derived. The possible counterexample has x1: (0,2,0); x2: (-1,0,0); x3: (1,0,0); x4-x5: (4,3,0)+epsilon_j, j=4,5; x6-x7: (-4,3,0)+epsilon_j, j=6,7; x8-x9: (0,-5,0)+epsilon_j, j=8,9. The binding determining hyperplanes are B_1={x1} with a hyperplane H1 that has normal (0,1,0); B_2={x3} with a hyperplane H2 that has normal (4/5,-3/5,0); B_3={x4} with a hyperplane H3 that has normal (4/5,3/5,0). In further research, we will either prove that the counter-example has mistakes in itself or updated current algorithm.