Maximin Fair Allocation

Authors

Taj Allamby

Affiliations

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The Problem

We wish to find the best way to allocate an m number group of indivisible items to an n number group of individuals (called agents). For the purpose of our research, we worked with 3 agents.

Why do we care?

Solving this problem helps us find the best way to fairly divide items up between a group of people. For example, let's say you and two friends were presented with a group of items, and these items can't be split up. Using these methods, you all can easily find the best way to divide everything up between you all, making sure no one is left feeling cheated or like they got the "worst end of the stick."

Definitions

- Proportional Share
  - This is 1/*n* of the total value of all items, with respect to the agents' valuation function.
  - Abbreviated to PS
- Maximin Share
  - The maximum over all partitions into n bundles, of the smallest value of a bundle under the agent's valuation function.
  - Abbreviated to MMS
  - An MMS-allocation is an allocation in which every agent receives at least their MMS.

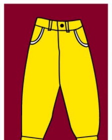
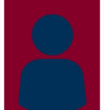
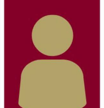
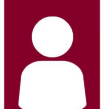
Solutions

We had two different methods we explored for finding the best MMS-allocation: *Exhaustive Search* and *Atomic Exhaustive Search.*

- Exhaustive Search
  - Baseline for the research.
  - For all possible allocations A = (A1, A2, . . . , An), we define ρA to be mini∈N (vi(Ai) M M Si). This tries all possible allocations and selects an A with the highest value of ρA.
  - Guarantees a value for ρ, making it optimal.
  - Drawbacks
    - Not clear to analyze what value of ρ is being guaranteed
    - Runs in exponential time to m, as it goes over all n^-m possibilities.
- Atomic Exhaustive Search
  - Partitions items into n^n atomic bundles, where there is a possibility that some of the bundles may be empty.
  - Each of these atomic bundles is an intersection between one bundle from each agent.
  - Each atomic bundle is treated one individual bundle set.

Lets look at an example!

Three GA Tech students, Navy, Gold and White, are given six items and told to give a **numerical score** to each one: a pair of pants, a teddy bear, a pen, a soccer ball, a lava lamp, and a car. The table below shows these scores and placements.

						
	28	15	10	33	35	52
	28	21	1	57	5	43
	17	1	66	25	79	9

Exhaustive Search Formula

Given n is the number of agents and m is the number of items.
v_j^i is the value of item j for player i, where $i = 1 \to n$ and $j = 1 \to m$
There exists an assignment such that each agent receives $\frac{11}{12}$ of their MMS.
$z_j^i = 1$ if item j is given to agent i, and 0 otherwise.
We can now get $\sum_{j=1\to m} z_j^i \times v_j^i$; the bundle value for player i.
Assignment Constraint: $\max_z \min_i \frac{\text{Assignment}_i(z)}{\text{MMS}(i)} \leq \frac{11}{12}$
MMS(i)= $\max_y$assignment $\min_{i'} \sum_j^n v_j^{i'} \times y_j^{i'}$
$\sum_{i=1}^3 y_j^i = 1$ where $y_j^i \epsilon \{0,1\}$

MMS Proof

We assume that n is 3, and that j is $1 \leq r \leq 3$, $1 \leq s \leq 3$, and/or $1 \leq t \leq 3$. We wish to prove that MMS$_i$ is greater than or equal to 1.

x_j^i : value of item j to agent i
From the given, we get $x_{r,s,t}^i$: value of atomic bundles
For each player, we can now assign a number to each of r, s, and t.
When finding the value of a players particular bundle, we add all of the possible 3-number combinations for r, s, and t in regards to $x_{r,s,t}^i$.
For example, we can have $x_{1,1,1}^i + x_{1,1,2}^i + x_{1,1,3}^i + x_{1,2,1}^i + x_{1,2,2}^i + x_{1,2,3}^i + x_{1,3,1}^i + x_{1,3,2}^i + x_{1,3,3}^i$
Given that each bundle has a positive value of 0 or higher, with at least one bundle having a value of at least 1, we can infer that the summation of these bundles would be greater than 1.
So now we have $x_{1,1,1}^i + x_{1,1,2}^i + x_{1,1,3}^i + x_{1,2,1}^i + x_{1,2,2}^i + x_{1,2,3}^i + x_{1,3,1}^i + x_{1,3,2}^i + x_{1,3,3}^i \geq 1$
Each agent can now create 3 of these bundles, one for each of the 3 number of players. The lowest of these 3 bundles becomes that player's MMS.
So, we have that MMS≥ 1.

Atomic Exhaustive Search Formula

Given MMS partitions for each player:

$\min_x z$ and MMS$_1 \geq 1$
$\forall$ allocation (A_1, A_2, A_3):
Either: $\sum_{j\epsilon A_1} x_j^1 \leq z$ or $\sum_{j\epsilon A_2} x_j^2 \leq z$ or $\sum_{j\epsilon A_3} x_j^3 \leq z$
We can then write that $a_1 + a_2 + a_3 = 1$ so that $a_i \epsilon \{0,1\}$

With this, we can rewrite our summations.
$\forall$ allocation (A_1, A_2, A_3):
Either: $\sum_{j\epsilon A_1} x_j^1 \leq z + (1-a_1) \times 10$ or $\sum_{j\epsilon A_2} x_j^2 \leq z + (1-a_2) \times 10$ or $\sum_{j\epsilon A_3} x_j^3 \leq z + (1-a_3) \times 10$

{A,E}∩{A,B,C}∩{A,B,D,F}={A}	{B,C,D}∩{A,B,C}∩{A,B,D,F}={B}	{F}∩{A,B,C}∩{A,B,D,F}=∅
{A,E}∩{A,B,C}∩{C}=∅	{B,C,D}∩{A,B,C}∩{C}={C}	{F}∩{A,B,C}∩{C}=∅
{A,E}∩{A,B,C}∩{E}=∅	{B,C,D}∩{A,B,C}∩{E}=∅	{F}∩{A,B,C}∩{E}=∅
{A,E}∩{D}∩{A,B,D,F}=∅	{B,C,D}∩{D}∩{A,B,D,F}={D}	{F}∩{D}∩{A,B,D,F}=∅
{A,E}∩{D}∩{C}=∅	{B,C,D}∩{D}∩{C}=∅	{F}∩{D}∩{C}=∅
{A,E}∩{D}∩{E}=∅	{B,C,D}∩{D}∩{E}=∅	{F}∩{D}∩{E}=∅
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{A,E}∩{E,F}∩{C}=∅	{B,C,D}∩{E,F}∩{C}=∅	{F}∩{E,F}∩{C}=∅
{A,E}∩{E,F}∩{E}={E}	{B,C,D}∩{E,F}∩{E}=∅	{F}∩{E,F}∩{E}=∅

Want more info? Scan here for the paper!