# Comparative Analysis of Bisection-Based Methods for Solving Chance Constrained Programs

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## Introduction

- Chance Constrained Program(CCP) : Find the best decision whose probability of violating the uncertainty constraints is within a preset risk level

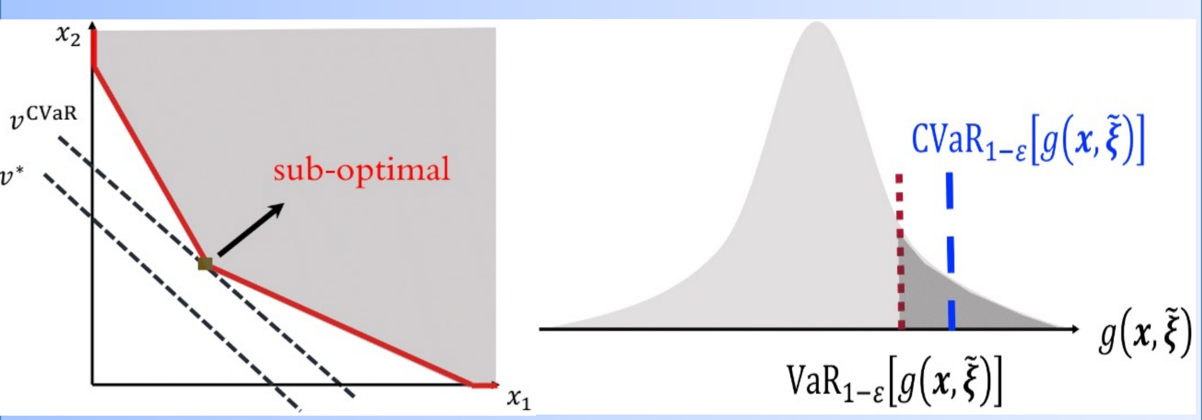
$$v^* = \min_{x \epsilon X}\{c^\top x: \mathbb{P}(\tilde{\xi}: g(x,\tilde{\xi}) \le 0) \ge 1-\varepsilon\}$$

- Applications of CCP:

Portfolio Selection    Wireless Networks

- One difficulty of CCP: The feasible region is non convex
- CVaR is known as the most effective convex approximation method

$$v^{\mathrm{CVaR}} = \min_{x \epsilon X}\{c^\top x: \mathrm{CVaR}_{(1-\varepsilon)}[g(x,\tilde{\xi})] \le 0$$




## Summary of Contributions

- New approximation methods are proposed to improve CVaR approximation
- Compare with DC approximation

**Methods:**

**ALSO-X Sharp**: Improvement by adjusting the upper bound of the CCP

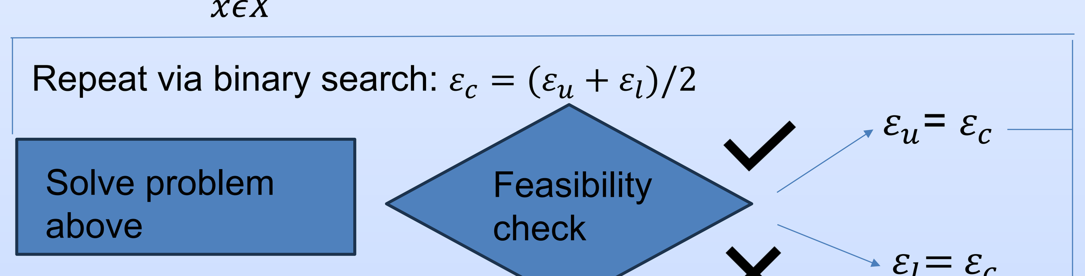
**Bisection-based Epsilon:** Improvement by adjusting the epsilon of the CCP

**Bisection-based RHS**: Improvement by adjusting the right-hand-side of the CCP

## Approximation Methods

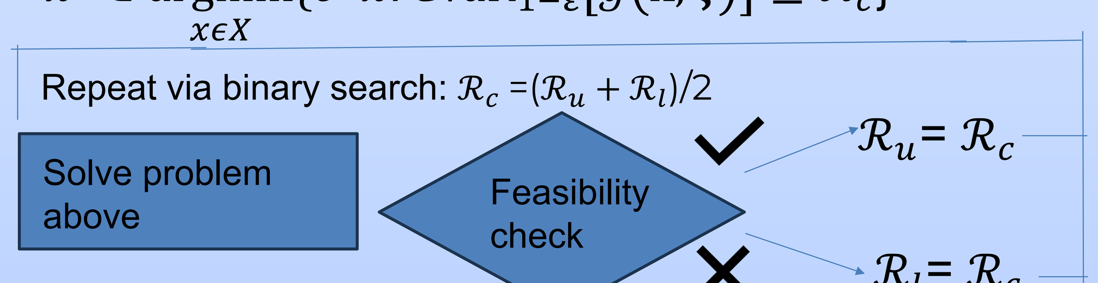
**Bisection-based Epsilon:**

$$x^* \in \operatorname*{argmin}_{x\epsilon X}\{c^T x: \mathrm{CVaR}_{1-\varepsilon_c}[g(x,\tilde{\xi})] \le 0\}$$



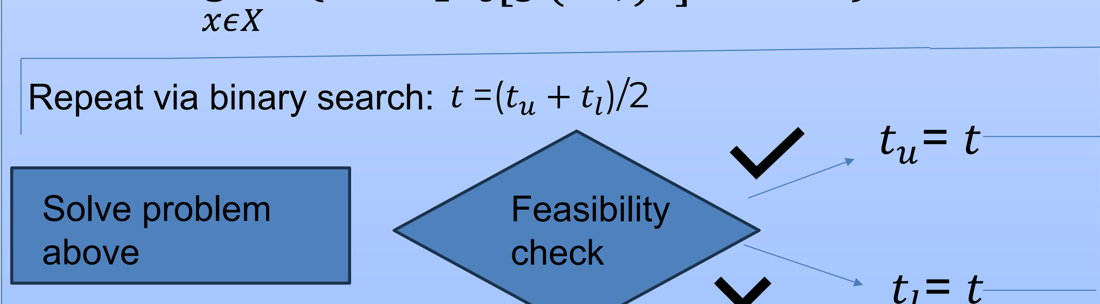

**Bisection-based RHS**:

$$x^* \in \operatorname*{argmin}_{x\epsilon X}\{c^T x: \mathrm{CVaR}_{1-\varepsilon}[g(x,\tilde{\xi})] \le \mathcal{R}_c\}$$




**ALSO-X #**:

$$x^* \in \operatorname*{argmin}_{x\epsilon X}\{\mathrm{CVaR}_{1-\varepsilon}[g(x,\tilde{\xi}):]\, c^T x \le t\}$$




## Numerical Comparison

instance Name 1-7-5-3000-5

| | CVAR | | ALSO-X # | | | Bisection-based-RHS | | | Bisection-based-Epsilon | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Epsilon value | Objective Value | Time (s) | Objective Value | Time (s) | Improvement (%) | Objective Value | Time (s) | Improvement (%) | Objective Value | Time (s) | Improvement (%) |
| 0.05 | -17519.71 | 0.11 s | -17627.19 | 20.98 | 0.61% | -17626.27 | 30.95 s | 0.61% | -17634.3 | 13.84 s | 0.65% |
| 0.1 | -17600.76 | 0.16 s | -17731.37 | 21.05s | 0.74% | -17731.21 | 31.40 s | 0.74% | -17750.89 | 14.34 s | 0.85% |
| 0.15 | -17657.97 | 0.18 s | -17824.63 | 21.41s | 0.94% | -17824.5 | 31.60 s | 0.94% | -17828.18 | 14.13 s | 0.96% |
| 0.2 | -17706.8 | 0.20 s | -17895.74 | 21.35 s | 1.07% | -17895.78 | 32.66 s | 1.07% | -17898.95 | 15.18 s | 1.09% |
| 0.25 | -17749.94 | 0.20 s | -17961.96 | 21.29s | 1.19% | -17962.12 | 32.87 s | 1.20% | -17965.65 | 15.33 s | 1.22% |

instance Name 1-7-5-3000-2

| | CVAR | | ALSO-X# | | | Bisection-based-RHS | | | Bisection-based-Epsilon | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Epsilon value | Objective value | Time(s) | Objective value | Time(s) | % Improvement | Objective value | Time(s) | % Improvement | Objective value | Time(s) | % Improvement |
| 0.05 | -17511.49 | 0.09 s | -17629.33 | 21.06s | 0.67% | -17629.16 | 31.58 s | 0.67% | -17631.73 | 14.05 s | 0.69% |
| 0.1 | -17596.69 | 0.15 s | -17736.99 | 21.20 s | 0.80% | -17736.99 | 31.73 s | 0.80% | -17744.41 | 14.68 s | 0.84% |
| 0.15 | -17656.29 | 0.18 s | -17822.83 | 21.41s | 0.94% | -17822.77 | 33.77 s | 0.94% | -17822.23 | 14.97 s | 0.94% |
| 0.2 | -17705.5 | 0.17 s | -17896.01 | 21.32s | 1.08% | -17896.15 | 31.72 s | 1.08% | -17900.75 | 15.31 s | 1.10% |
| 0.25 | -17749.26 | 0.20 s | -17968.16 | 21.47 s | 1.23% | -17968.39 | 32.75 s | 1.23% | -17962.36 | 15.55 s | 1.20% |

instance Name 1-7-5-3000-3

| | CVAR | | ALSO-X# | | | Bisection-based-RHS | | | Bisection-based-Epsilon | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Epsilon value | Objective value | Time(s) | Objective value | Time(s) | % Improvement | Objective value | Time(s) | % Improvement | Objective value | Time(s) | % Improvement |
| 0.05 | -17512.7 | 0.59s | -17608.9 | 21.58 | 0.55% | -17609.44 | 32.47 s | 0.55% | -17623.47 | 14.86s | 0.63% |
| 0.1 | -17587.87 | 0.26s | -17720.15 | 22.40s | 0.75% | -17720.75 | 32.58 s | 0.76% | -17731.07 | 14.31s | 0.81% |
| 0.15 | -17648.55 | 0.15s | -17818.74 | 21.73s | 0.96% | -17818.58 | 32.76 s | 0.96% | -17820.15 | 14.59 s | 0.97% |
| 0.2 | -17698.94 | 0.15s | -17891.78 | 22.22s | 1.09% | -17891.5 | 31.36 s | 1.09% | -17889.61 | 14.80 s | 1.08% |
| 0.25 | -17741.75 | 0.19s | -17954.14 | 22.59s | 1.20% | -17954.21 | 32.48 s | 1.20% | -17957.43 | 15.95 s | 1.22% |

instance Name 1-7-5-3000-4

| | CVAR | | ALSO-X # | | | Bisection-based-RHS | | | Bisection-based-Epsilon | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Epsilon value | Objective Value | Time (s) | Objective Value | Time (s) | Improvement (%) | Objective Value | Time (s) | Improvement (%) | Objective Value | Time (s) | Improvement (%) |
| 0.05 | -17495.494 | 0.16s | -17596.83 | 21.63s | 0.58% | -17597.37 | 31.76 s | 0.58% | -17606.92 | 15.07 s | 0.64% |
| 0.1 | -17578.57 | 0.19s | -17728.45 | 21.3s | 0.85% | -17728.49 | 32.06 s | 0.85% | -17748.885 | 13.98 s | 0.97% |
| 0.15 | -17643.55 | 0.32s | -17828.14 | 21.92 s | 1.05% | -17828.44 | 31.70 s | 1.05% | -17827.14 | 14.73 s | 1.04% |
| 0.2 | -17697.29 | 0.24s | -17901.14 | 21.28 | 1.15% | -17901.03 | 31.24 s | 1.15% | -17902.49 | 15.09 s | 1.16% |
| 0.25 | -17741.93 | 0.20s | -17964.87 | 22.04 s | 1.26% | -17964.99 | 31.61 s | 1.26% | -17964.65 | 15.48 s | 1.26% |

## Results

- RED= Most optimal objective value.   YELLOW= Quickest running time
- All three methods improve CVaR approximation
- About 90% of the time, **Bisection-based Epsilon** is the best

## DC approximation

- These 4 methods can serve as the initial solutions for DC approximation, which is defined as

$$\min_{\{x\in X\}} \{c^\top x: \tfrac{1}{\hat{\varepsilon}}\mathbb{E}[g(x,\tilde{\xi}) + \hat{\varepsilon}]_+ - \tfrac{1}{\hat{\varepsilon}}\mathbb{E}[g(x,\tilde{\xi})]_+ \le 0\}$$

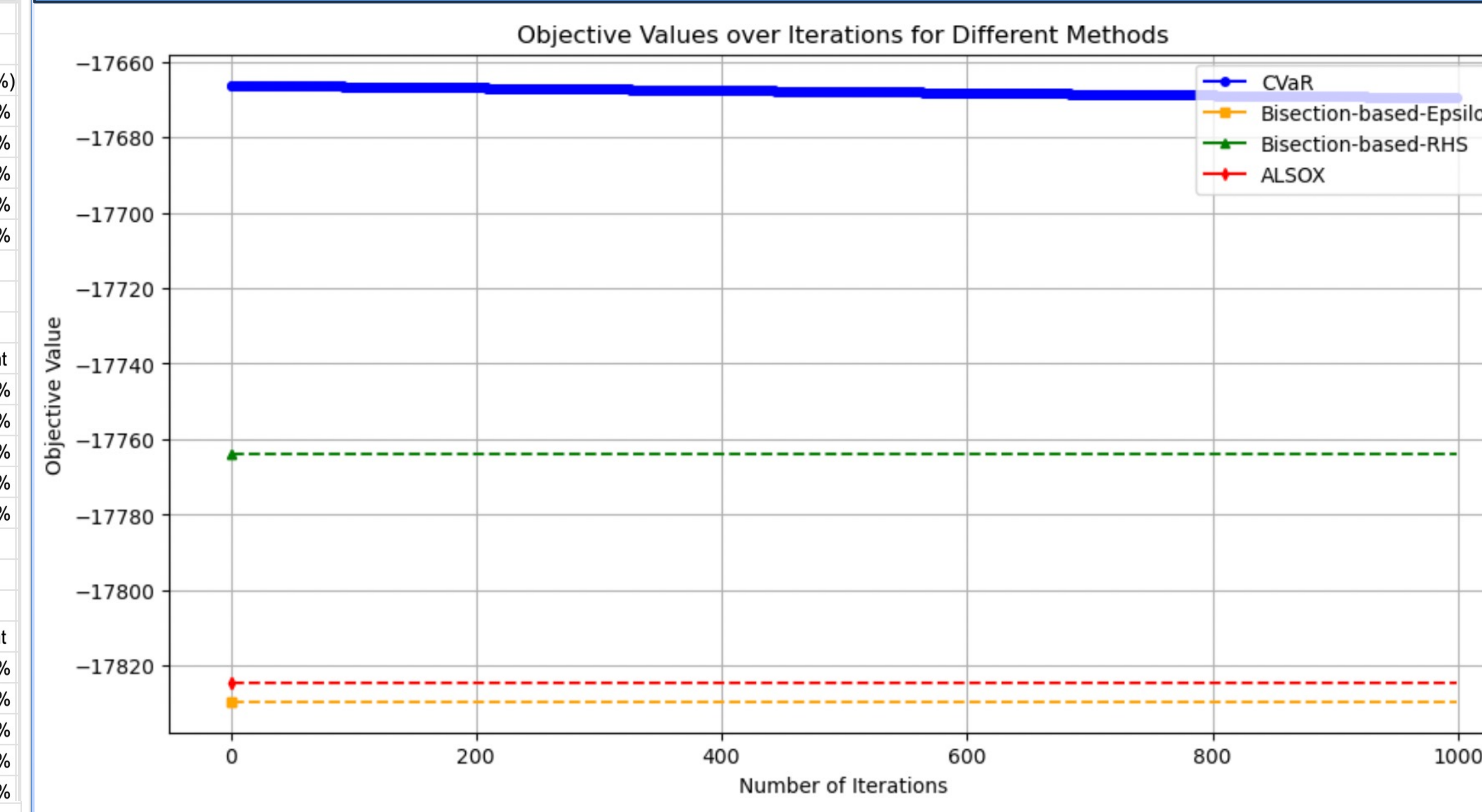
- After taking the derivative of the inner convex function, we follow a sub gradient-based algorithm to solve the DC approximation

$$\mathbb{E}[g(x,\tilde{\xi})]_+ \ge \mathbb{E}[g(x^{(o)},\tilde{\xi})]_+ + \partial_{x^{(o)}}\mathbb{E}[g(x^{(o)},\tilde{\xi})]_+[x - x^{(o)}].$$

## Summary

- Cvar approximation is employed to evaluate the potential losses in financial decision-making, serving as a critical tool for risk management. Investigating other methods that can improve CVaR, such as **Bisection-based Epsilon**, provides a more accurate estimate of potential extreme losses
- Under type ∞- Wasserstein ambiguity set, **Bisection-based Epsilon** improves the CVaR approximation results 70% of the time across different epsilons and thetas over varying instances

## Objective Values over Iterations for Different Methods




**Initial Solutions**

CVaR Approximation
Bisection-based Epsilon
Bisection- based RHS
ALSO-X#

**Iterations**

Instances were tested across varying numbers of iterations

**Results**

DC is deemed unnecessary for improving these initial solutions

## Distributionally Robust Chance Constrained Programs

DRCCP's seek to determine optimal results that handle uncertain constraints and parameters, while taking into account all possible distributions in the Wasserstein ambiguity set.

$$v^* = \min_{x\epsilon X}\{c^\top x: \inf_{\mathbb{P}\epsilon\mathcal{P}_\infty} \mathbb{P}(\tilde{\xi}: g(x,\tilde{\xi}) \le 0) \ge 1-\varepsilon\}$$